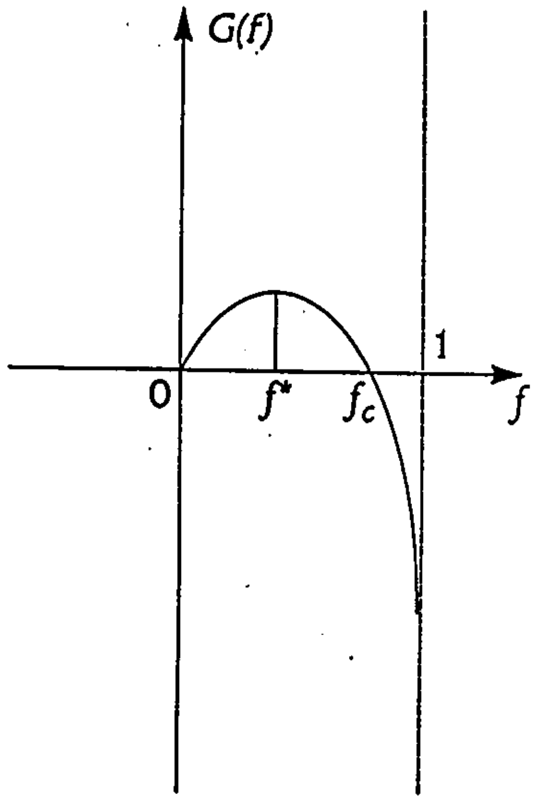


Figure 1

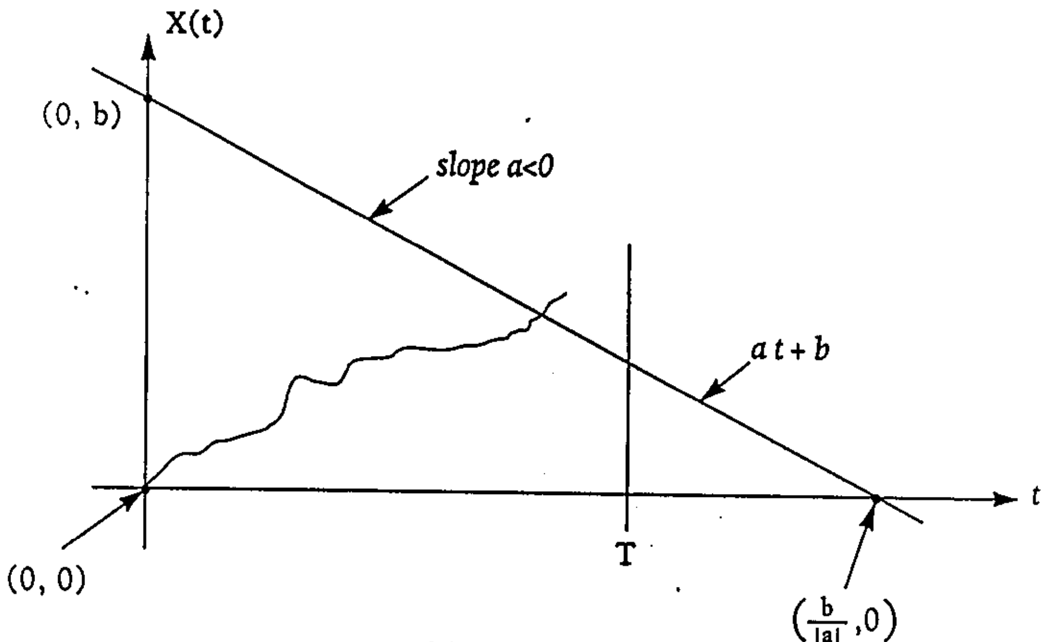


Figure 2

Figure 3

Betting Log · Type 2 sports

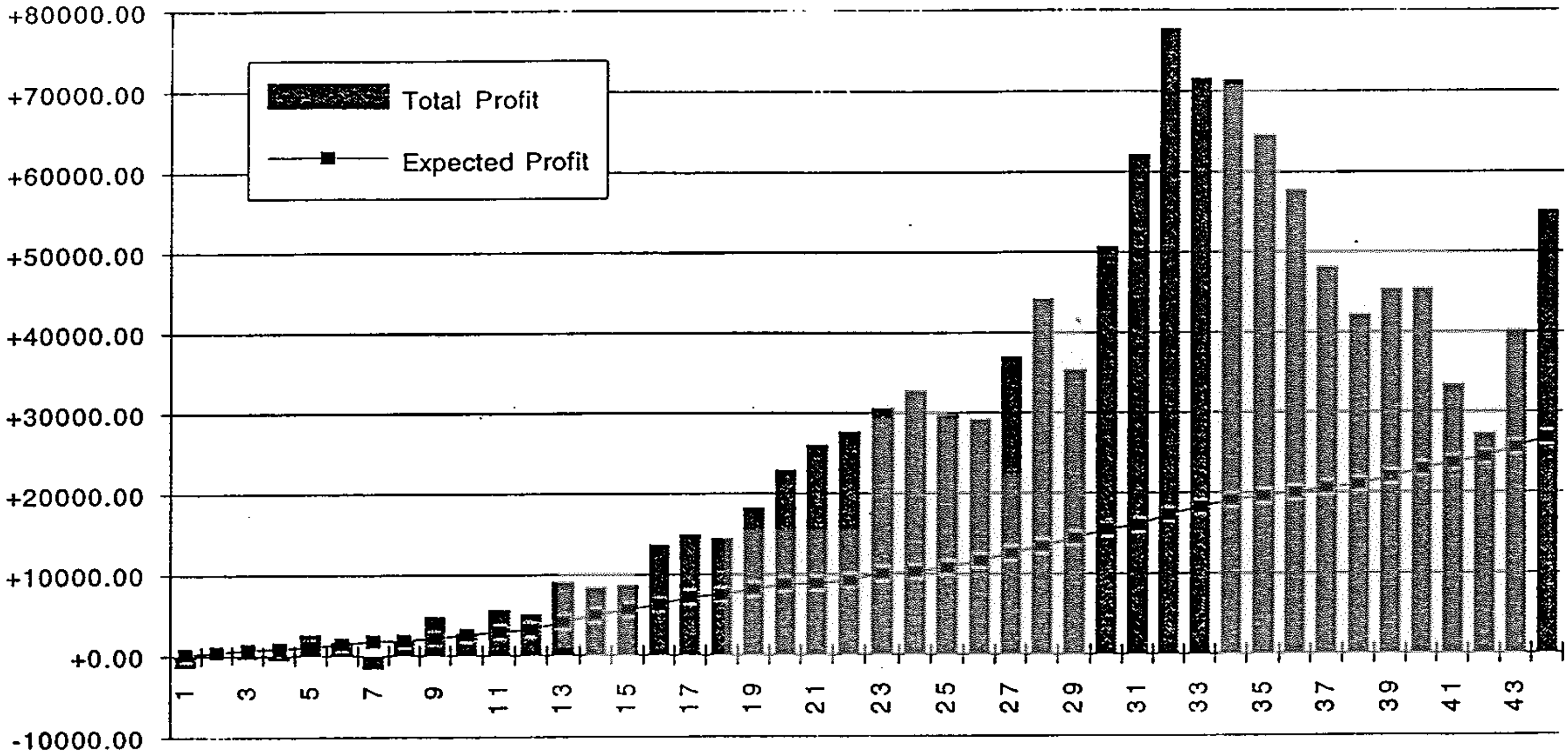


Figure 4 Betting Log Type 1 sports

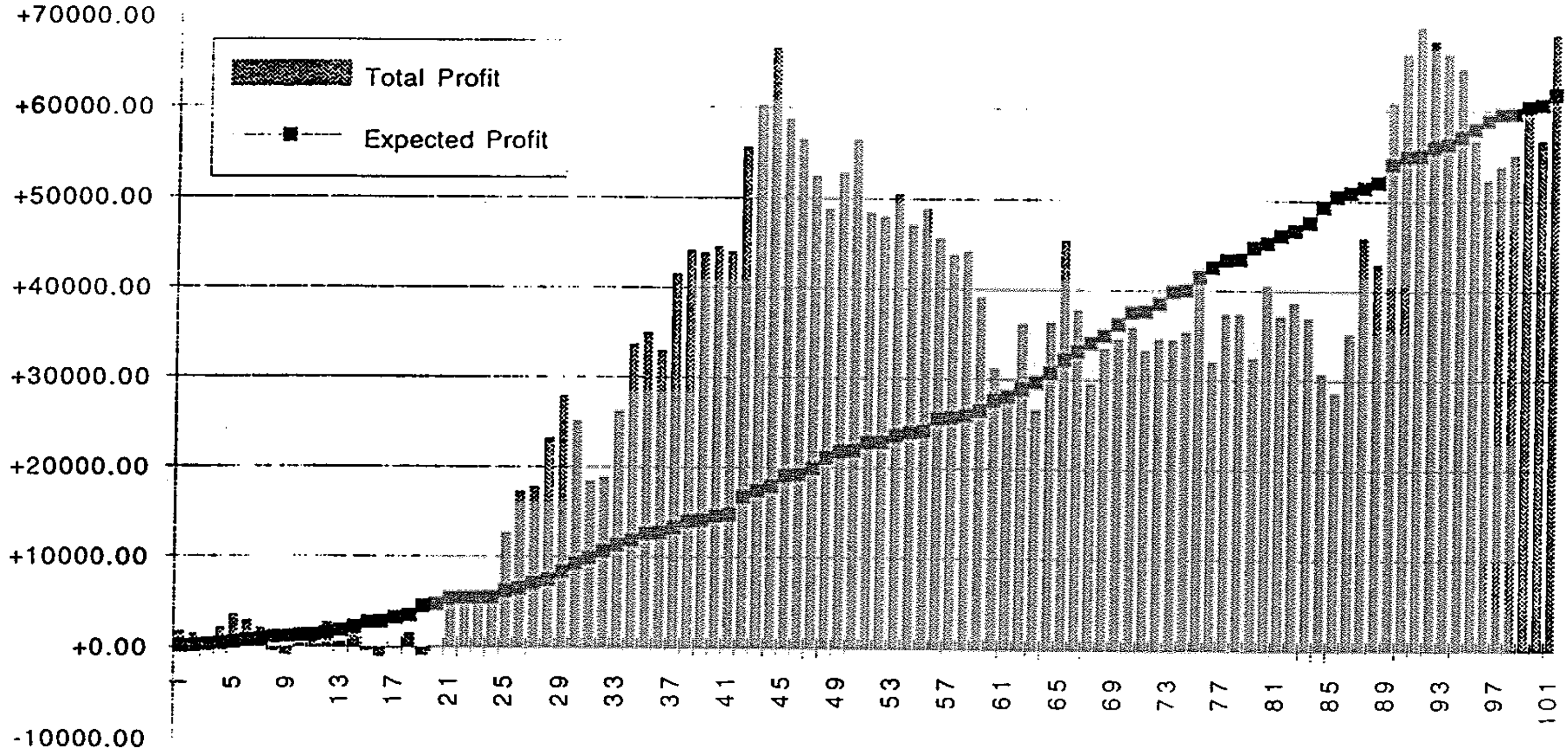
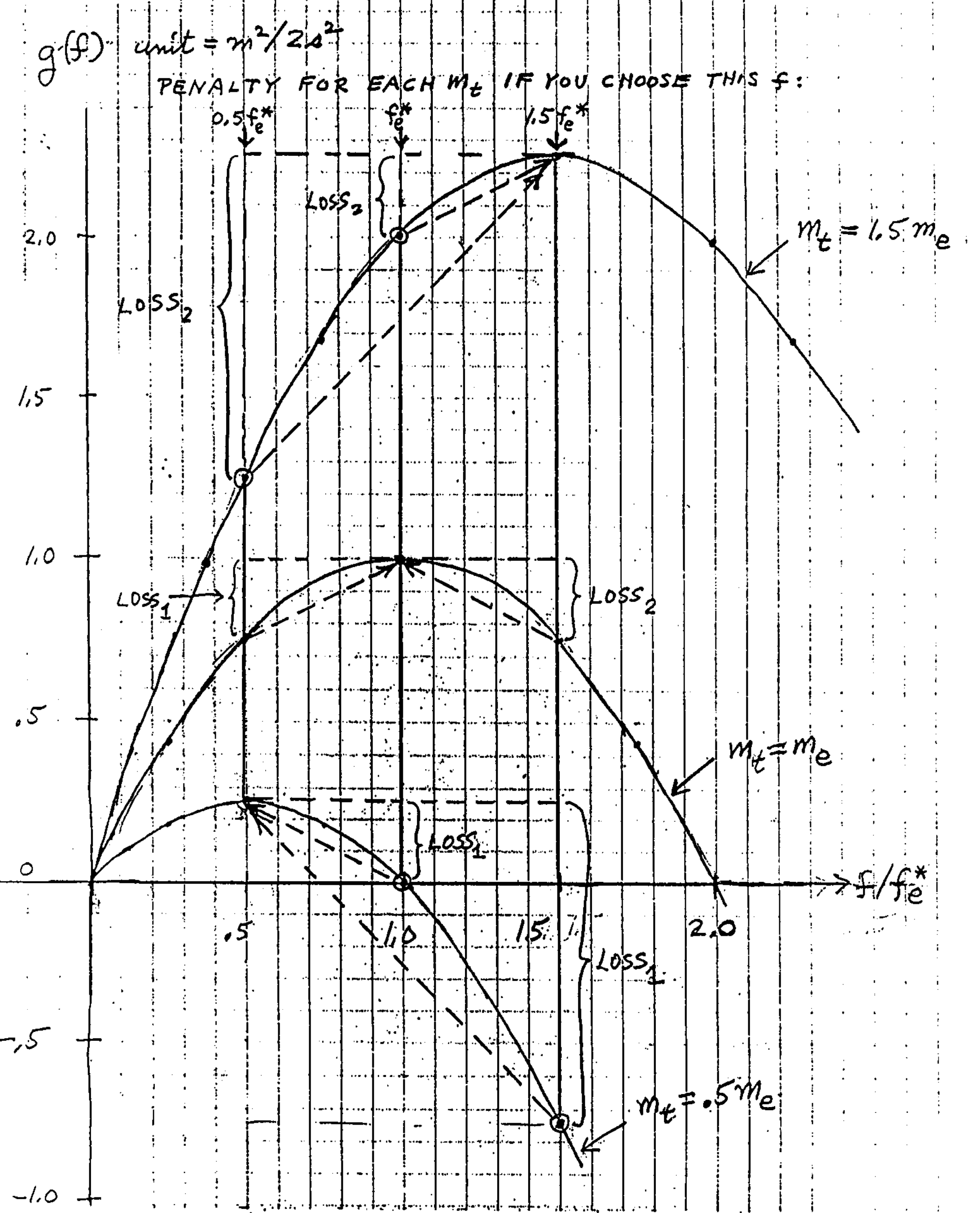


FIGURE 5. Penalties for choosing $f = f_e \neq f^* = f_t$



APPENDIX I. Integrals for deriving moments of E_∞

$$
\begin{aligned}
I_0\left(a^2, b^2\right) &= \int_0^\infty \exp\left[-\left(a^2x^2 + b^2/x^2\right)\right] dx \\
I_n\left(a^2, b^2\right) &= \int_0^\infty x^n \exp\left[-\left(a^2x^2 + b^2/x^2\right)\right] dx
\end{aligned}
$$

Given I_0 find I_2

$$
\begin{aligned}
I_0\left(a^2, b^2\right) &= \int_0^\infty \exp\left[-\left(a^2x^2 + b^2/x^2\right)\right] dx \\
&= -\int_\infty^0 \exp\left[-\left(a^2/u^2 + b^2u^2\right)\right]\left(-du/u^2\right)
\end{aligned}
$$

where $x = 1/u$ and $dx = -du/u^2$ so

$$
\begin{aligned}
I_0\left(a^2, b^2\right) &= \int_0^\infty x^{-2} \exp\left[-\left(b^2x^2 + a^2/x^2\right)\right] \\
&= I_{-2}\left(b^2, a^2\right)
\end{aligned}
$$

$$
\text{(A1)} \qquad \text{hence} \quad I_{-2}\left(a^2, b^2\right) = I_0\left(b^2, a^2\right) = \frac{\sqrt{\pi}}{2 \mid b \mid} e^{-2|ab|}
$$

$$
I_0 = \int_0^\infty \exp\left[-\left(a^2x^2 + b^2/x^2\right)\right] dx = U \cdot V \Big|_0^\infty - \int_0^\infty V dU
$$

where $U = \exp[\,\cdot\,]$, $dV = dx$, $dU = (\exp[\,\cdot\,])(-2a^2x + 2b^2x^{-3})$ and $V = x$ so

$$
\begin{aligned}
I_0 &= \exp\left[-\left(a^2x^2 + b^2/x^2\right)\right] \cdot x \Big|_0^\infty - \int_0^\infty \left(-2a^2x^2 + 2b^2/x^2\right) \exp\left[-\left(a^2x^2 + b^2/x^2\right)\right] dx \\
&= 2a^2 I_2\left(a^2, b^2\right) - 2b^2 I_{-2}\left(a^2, b^2\right)
\end{aligned}
$$

Hence:

$$
I_0\left(a^2, b^2\right) = 2a^2 I_2\left(a^2, b^2\right) - 2b^2 I_{-2}\left(a^2, b^2\right)
$$

and $I_{-2}(a^2, b^2) = I_0(b^2, a^2)$ so substituting and solving for I_2 gives

$$I_2\left(a^2, b^2\right) = \frac{1}{2a^2}\left\{I_0\left(a^2, b^2\right) + 2b^2 I_0\left(b^2, a^2\right)\right\}$$

Comments.
1. We can solve for all even n by using I_0, I_{-2} and I_2, and integration by parts.
2. We can use the indefinite integral J_0 corresponding to I_0, and the previous methods, to solve for J_{-2}, J_2, and then for all even n. Since

$$\begin{aligned}
I_0\left(a^2, b^2\right) &= \frac{\sqrt{\pi}}{2 \mid a \mid} e^{-2|ab|} \qquad \text{then} \\
I_{-2}\left(b^2, a^2\right) &= \frac{\sqrt{\pi}}{2 \mid b \mid} e^{-2|ab|} \qquad \text{and} \\
I_2\left(a^2, b^2\right) &= \frac{1}{2a^2}\left\{\frac{\sqrt{\pi}}{2 \mid a \mid} + 2b^2 \frac{\sqrt{\pi}}{2 \mid b \mid}\right\} e^{-2|ab|} \\
&= \frac{\sqrt{\pi}}{4a^2} e^{-2|ab|} \left\{1/ \mid a \mid + 2 \mid b \mid\right\}
\end{aligned}$$

APPENDIX II. Derivation of Formula (3.1)

This is based on a note from Howard Tucker. Any errors are mine.
*From the paper by Paranjape & Park, if $X(t)$ is standard Brownian motion, if $a \neq 0$, $b > 0$,

$$P(X(t) \leq at + b,\ 0 \leq t \leq T \mid X(T) = s)$$

$$= \begin{cases} 1 - exp\left\{-\frac{2b}{T}(aT + B - s)\right\} & \text{if } s \leq aT + b \\ 0 \text{ if } s > aT + B. \end{cases}$$

Write this as:

$$P(X(t) \leq at + b,\ 0 \leq t \leq T \mid X(T)) \overset{a.s.}{=}$$

$$= (1 - exp\{-2b(aT + b - X(T))\frac{1}{T}\} \quad \text{if} \quad X(T) \leq aT + b.$$

Taking expectations of both sides of the above, we get

$$P(X(t) \leq at + b,\ 0 \leq t \leq T)$$

$$= \int_{-\infty}^{aT+b} \left(1 - e^{-2b(aT+b-s)1/T}\right) \frac{1}{\sqrt{2\pi T}} e^{-s^2/2T} ds$$

$$= \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{aT+b} e^{-s^2/2T} ds - \frac{e^{-2ab}}{\sqrt{2\pi T}} \int_{-\infty}^{aT+b} e^{-(s-2b)^2/2T} ds$$

Hence

(I) $\quad P(X \text{ going above line } at + b \text{ during } [O, T]) = 1 -$ previous probability

$$= \frac{1}{\sqrt{2\pi T}} \int_{aT+b}^{\infty} e^{-s^2/2T} ds + e^{-2ab} \cdot \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{aT-b} e^{-u^2/2T} du, \text{ where } u = s - 2b.$$

Now, when $a = 0$, $b > 0$,

$$P\left[\sup_{0 \leq t \leq T} X(t) \geq b\right] = \sqrt{\tfrac{2}{\pi T}} \int_{b}^{\infty} e^{-v^2/2T} dv,$$

which agrees with a known formula (see, e.g., page 261 of Tucker [1967]). When $T \to \infty$, since $\sqrt{T}/T \to 0$ and $\sqrt{T} = s.d.\,\text{of}\, X(T)$, the first integral $\to 0$, the second integral $\to 1$, and $P(X\,\text{ever rises above line}\, at + b) = e^{-2ab}$.

*The theorem it comes from is due to Sten Malmquist: On Certain Confidence Countours for Distribution Functions, Ann. Math. Stat., 25 (1954), pp 523-533. This theorem is stated in S.R. Paranjape and C. Park: Distribution of the Supremum of the Two-Parameter Yeh-Wiener Process on the Boundary, J. Appl. Prob. Vol. 10 (1973).

Letting $\propto = a\sqrt{T}$, $\beta = b/\sqrt{T}$, formula I becomes

$$P(\cdot) = N(-\propto - \beta) + e^{-2\propto\beta} N(\propto - \beta) \text{ where } \propto, \beta > 0 \quad \text{or}$$

$$\text{(II)} \quad P(X(t) \leq at + b,\, 0 \leq t \leq T) = 1 - P(\cdot) = N(\propto + \beta) - e^{-2\propto\beta} N(\propto - \beta)$$

for the probability the line is never surpassed. This follows from:

$$\frac{1}{\sqrt{2\pi T}} \int_{aT+b}^{\infty} e^{-s^2/2T} ds = \frac{1}{\sqrt{2\pi}} \int_{a\sqrt{T}+b/\sqrt{T}}^{\infty} e^{-x^2/2} dx = N(-\propto - \beta)$$

$$\text{and} \qquad \frac{1}{\sqrt{2\pi T}} \int_{-\infty}^{aT-b} e^{-u^2/2T} du = N(\propto - \beta)$$

where $s = aT + b$ and $x = s/\sqrt{T} = a\sqrt{T} + b/\sqrt{T}$
and $\propto = a\sqrt{T}$ and $\beta = b/\sqrt{T}$.

The formula becomes:

$$P(\sup[X(t) - (at + b)] \geq 0 : 0 \leq t \leq T)$$

$$\begin{aligned} &= N(-\propto - \beta) + e^{-2ab} N(\propto - \beta) \\ &= N(-\propto - \beta) + e^{-2\propto\beta} N(\propto - \beta) \quad \propto, \beta > 0 \end{aligned}$$

Observe that $P(\cdot) < N(-\propto - \beta) + N(\propto - \beta)$

$$\begin{aligned} &= \{1 - N(\propto + \beta)\} + N(\propto - \beta) \\ &= \int_{-\infty}^{\propto - \beta} \propto (x) dx + \int_{\propto + \beta}^{\infty} \propto (x) dx < 1 \end{aligned}$$

as it should be.

Appendix III. Expected time to reach goal

Reference: Handbook of Mathematical Functions, Abramowitz and Stegun, Editors, N.B.S. Applied Math. Series 55, June 1964.
p.304, 7.4.33 gives with $erf z \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$ the integral:

$$(1) \quad \int \exp\{-(a^2x^2+b^2/x^2)\}dx = \frac{\sqrt{\pi}}{4a}\left[e^{2ab}erf(ax+b/x)+e^{-2ab}erf(ax-b/x)\right]+C,$$

$a \neq 0$.

Now the left side is > 0 so for real a, we require $a > 0$ otherwise the right side is < 0, a contradiction.

We also note that p.302, 7.4.3. gives

$$(2) \qquad \int_0^\infty \exp\{-(at^2+b/t^2)\}dt = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}}$$

with $\Re a > 0$, $\Re b > 0$.

To check (2) v. (1), suppose in (1) $a > 0$, $b > 0$ and find

$$\lim_{x \to 0} \text{ and } \lim_{x \to \infty} \text{ of } erf(ax+b/x) \text{ and } erf(ax-b/x)$$

$$\lim_{x \downarrow 0+}(ax+b/x) = +\infty \qquad \lim_{x \downarrow 0+}(ax-b/x) = -\infty$$

$$\lim_{x \to \infty}(ax+b/x) = +\infty \qquad \lim_{x \to \infty}(ax-b/x) = +\infty$$

(1) becomes

$$\begin{aligned} \frac{\sqrt{\pi}}{4a}e^{-2ab}\left[erf(\infty)-erf(-\infty)\right] &= \frac{\sqrt{\pi}}{4a}e^{-2ab}2erf(\infty) \\ &= \frac{\sqrt{\pi}}{2a}e^{-2ab} \end{aligned}$$

since we know $erf(\infty) = 1$.

In (2) replace a by a^2, b by b^2 to get

$$I_0(a^2,b^2) \equiv \int_0^\infty \exp\{-(a^2t^2+b^2/t^2)\}dt = \frac{1}{2}\frac{\sqrt{\pi}}{|a|}e^{-2|ab|}$$

which is the same.

Note: if we choose the lower limit of integration to be 0 in (1), then we can find C:

$$\begin{aligned} 0 &= \int_0^0 \exp\{-(a^2x^2+b^2/x^2)\}dx \\ &= \frac{\sqrt{\pi}}{4a}\left[e^{2ab}erf(\infty)+e^{-2ab}erf(-\infty)\right]+C \\ &= \frac{\sqrt{\pi}}{4a}\left[e^{2ab}-e^{-2ab}\right]+C. \end{aligned}$$

Whence

$$\begin{aligned} (3) \qquad F(x) &\equiv \int_0^x \exp\{-(a^2x^2+b^2/x^2)\}dx \\ &= \frac{\sqrt{\pi}}{4a}\left\{e^{2ab}[erf(ax+b/x)-1]+e^{-2ab}[erf(ax-b/x)+1]\right\} \end{aligned}$$

To see how (3) might have been discovered, differentiate:

$$\begin{aligned} F'(x) &= \exp\{-(a^2x^2+b^2/x^2)\} \\ &= \frac{\sqrt{\pi}}{4a}\{e^{2ab}(a-b/x^2)erf'(ax+b/x)+e^{-2ab}(a+b/x^2)erf'(ax-b/x)\} \end{aligned}$$

Now $erf'(z)=\frac{2}{\sqrt{\pi}}\exp(-z^2)$ so

$$\begin{aligned} erf'(ax+b/x) &= \frac{2}{\sqrt{\pi}}\exp[-(ax+b/x)^2] \\ &= \frac{2}{\sqrt{\pi}}\exp\{-(a^2x^2+b^2/x^2+2ab)\} \\ &= \frac{2}{\sqrt{\pi}}e^{-2ab}\exp\{-(a^2x^2+b^2/x^2)\} \end{aligned}$$

and, setting $b \leftarrow -b$,

$$erf'(ax-b/x)=\frac{2}{\sqrt{\pi}}e^{2ab}\exp\{-(a^2x^2+b^2/x^2)\}$$

whence

$$\begin{aligned} F'(x) &= \frac{\sqrt{\pi}}{4a}\left\{\frac{2}{\sqrt{\pi}}(a-b/x^2)+\frac{2}{\sqrt{\pi}}(a+b/x^2)\right\}\exp\{-(a^2x^2+b^2/x^2)\} \\ &= \frac{1}{2a}\{2a\}\exp\{-(a^2x^2+b^2/x^2)\} \\ &= \exp\{-(a^2x^2+b^2/x^2)\} \end{aligned}$$

Case of interest: $a < 0$, $b > 0$
Expect:
$b > 0,\ a \leq 0 \ \Rightarrow\ F(T) \uparrow 1 \,\text{as}\, T \to \infty$
$b > 0,\ a > 0 \ \Rightarrow\ F(T) \uparrow c < 1 \,\text{as}\, T \to \infty$
If $b > 0,\ a = 0$:

$$F(T) = N(-\beta) + N(-\beta) = 2N(-b/\sqrt{T}) \uparrow 2N(0) = 1 \,\text{as}\, T \uparrow \infty.$$

Also, as expected $F(T) \uparrow 1$ as $b \downarrow 0$.
If $b > 0,\ a < 0$: See Below.
If $b > 0,\ a > 0$:

$$\begin{aligned} F(T) &= N(-a\sqrt{T} - b/\sqrt{T}) + e^{-2ab}N(a\sqrt{T} - b/\sqrt{T}) \to N(-\infty) + e^{-2ab}N(\infty) \\ &= e^{-2ab} < 1 \,\text{as}\, T \uparrow \infty. \qquad \text{This is correct.} \end{aligned}$$

If $b = 0$: $F(T) = N(-a\sqrt{T}) + N(a\sqrt{T}) = 1$. This is correct.

Let $F(T) = P(X(t) \geq at + b \,\text{for some}\, t,\ 0 \leq t \leq T)$ which equals $N(-\propto -\beta) + e^{-2ab}N(\propto -\beta)$ where $\propto = a\sqrt{T}$ and $\beta = b/\sqrt{T}$ so $ab = \propto \beta$; we assume $b > 0$ and $a < 0$ in which case $0 \leq F(T) \leq 1$ and $\lim_{T\to\infty} F(T) = 1$, $\lim_{T\to 0} F(T) = 0$; F is a probability distribution function:

$$\begin{aligned} \lim_{T\to 0} F(T) &= N(-\infty) + e^{-2ab}N(-\infty) = 0 \\ \lim_{T\to\infty} F(T) &= N(+\infty) + e^{-2ab}N(-\infty) = 1. \end{aligned}$$

The density function

$$\begin{aligned} f(T) &= F'(T) \\ &= \frac{\partial}{\partial T}(-\propto -\beta)N'(-\propto -\beta) + e^{-2ab}\frac{\partial}{\partial T}(\propto -\beta)N'(\propto -\beta) \end{aligned}$$

where

$$\frac{\partial \propto}{\partial T} = \frac{1}{2}aT^{-1/2} \qquad \frac{\partial \beta}{\partial T} = -\frac{1}{2}bT^{-3/2}$$

$$\begin{aligned} N'(-\propto -\beta) &= \frac{1}{\sqrt{2\pi}}e^{-(\propto+\beta)^2/2} = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(a^2T + b^2/T + 2ab)}{2}\right\} \\ N'(\propto -\beta) &= \frac{1}{\sqrt{2\pi}}e^{-(\propto-\beta)^2/2} = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(a^2T + b^2/T - 2ab)}{2}\right\} \end{aligned}$$

$$
\begin{aligned}
Tf(T) &= T\left(-\frac{1}{2}aT^{-1/2}+\frac{1}{2}bT^{-3/2}\right)\frac{1}{\sqrt{2\pi}}e^{-ab}\exp\left\{-\frac{(a^2T+b^2/T)}{2}\right\}\\
&+ Te^{-2ab}\left(\frac{1}{2}aT^{-1/2}+\frac{1}{2}bT^{-3/2}\right)\frac{1}{\sqrt{2\pi}}e^{ab}\exp\left\{-\frac{(a^2T+b^2/T)}{2}\right\}\\
&= \frac{e^{-ab}}{2\sqrt{2\pi}}\begin{bmatrix}(-aT^{+1/2}+bT^{-1/2})\exp\{-\frac{(a^2T+b^2/T)}{2}\}\\ +(aT^{+1/2}+bT^{-1/2})\exp\{-\frac{(a^2T+b^2/T)}{2}\}\end{bmatrix}
\end{aligned}
$$

The expected time to the goal is

$$
\begin{aligned}
E_\infty = \int_0^\infty Tf(T)dT &= \frac{be^{-ab}}{\sqrt{2\pi}}\int_0^\infty T^{-1/2}\exp\left\{-\frac{(a^2T+b^2/T)}{2}\right\}dT\\
\left.\begin{array}{r} T^{1/2}=x\\ T=x^2\\ dT=2xdx\end{array}\right\} &= \frac{2be^{-ab}}{\sqrt{2\pi}}\int_0^\infty \exp\left\{-\left[\left(\frac{a}{\pi 2}\right)^2x^2+\left(\frac{b}{\pi 2}\right)^2x^{-2}\right]\right\}dx\\
&= \frac{2be^{-ab}}{\sqrt{2\pi}}I_0\left(\left(\frac{a}{\sqrt{2}}\right)^2,\left(\frac{b}{\sqrt{2}}\right)^2\right)
\end{aligned}
$$

Now

$$
\begin{aligned}
I_0(a^2,b^2) &= \frac{\sqrt{\pi}}{2\mid a\mid}e^{-2|ab|} \quad \text{so}\\
I_0\left(\left(\frac{a}{\sqrt{2}}\right)^2,\left(\frac{b}{\sqrt{2}}\right)^2\right) &= \frac{\sqrt{\pi}}{\sqrt{2}\mid a\mid}e^{-|ab|} \quad \text{whence}\\
E_\infty &= \frac{2be^{-ab}}{\sqrt{2\pi}}\frac{\sqrt{\pi}}{\sqrt{2}\mid a\mid}e^{-|ab|} = \frac{b}{\mid a\mid},\ a<0,\ b>0.
\end{aligned}
$$

Note:

$$
F'(T) = \frac{be^{-ab}}{\sqrt{2\pi}}T^{-3/2}\exp\left\{\frac{-(a^2T+b^2/T)}{2}\right\} > 0
$$

for all a, e.g. $a<0$, so $F(T)$ is monotone increasing. Hence, since $\lim_{T\to\infty}F(T)=1$, $0\le F(T)\le 1$ for all T so we have more confidence in using the formula for $a<0$ too.

Check:	$E_\infty(a,b)\downarrow 0$ as $\downarrow -\infty$	yes
	$E_\infty(a,b)\uparrow$ as $b\uparrow$	yes

$$
\begin{array}{lll}
 & E_\infty(a,b) \uparrow \text{ as } \mid a \mid\downarrow & \text{yes} \\
\text{note} & \lim\limits_{a\downarrow 0^+} E_\infty(a,b) = +\infty & \text{as suspected}
\end{array}
$$

This leads us to believe that in a fair coin toss (fair means no drift) and a gambler with finite capital, the expected time to ruin is infinite.

This is correct. Feller give $D = z(a-z)$ as the duration of the game, where z is initial capital, ruin is at 0, and a is the goal. Then $\lim\limits_{a\to\infty} D(a) = +\infty$. NOTE: $E_\infty = b/\mid a \mid$ means the expected time is the same as the point where $aT + b$ crosses $X(t) = 0$. See Figure 2.

$$
\begin{aligned}
E_\infty &= b/\mid a \mid \qquad a = -m/s^2 \qquad b = ln\lambda \\
\lambda &= C/X_0 = \text{normalized goal} \\
m &= pln(1+f) + qln(1-f) \equiv g(f) \\
s^2 &= pq\{ln[(1+f)/(1-f)]\}^2 \\
\text{Kelly fraction } f^* &= p - q \qquad g(f^*) = pln2p + qln2q \\
\text{For } m > 0,\ E_\infty &= (ln\lambda)s^2/g(f)
\end{aligned}
$$

Now this is the expected time in variance units. However s^2 variance units = 1 trial so

$$
n(\lambda, f) \equiv \frac{E_\infty}{s^2} = \frac{ln\lambda}{g(f)} = \frac{ln\lambda}{m}
$$

is the expected number of trials.

$$
\begin{array}{ll}
\text{Checks:} & n(\lambda,f) \uparrow \text{ as } \lambda \uparrow \\
 & n(\lambda,f) \to \infty \text{ as } \lambda \to \infty \\
 & n(\lambda,f) \uparrow \text{ as } m \downarrow 0 \\
 & n(\lambda,f) \to \infty \text{ as } m \to 0
\end{array}
$$

Now $g(f)$ has unique maximum at $g(f^*)$ where $f^* = p - q$, the "Kelly fraction," therefore $n(\lambda, f)$ has a unique minimum for $f = f^*$. Hence f^* reaches a fixed goal in least expected time in this, the continuous case, so we must be asymptotically close to least expected time in the discrete case, which this approximates increasing by well in the sense of the CLT (Central Limit Theorem) and its special case, the normal approximation to the binomial distribution. The difference here is the trials are asymmetric. The positive and negative step sizes are unequal.

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